$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y)$$

MATH 162A Review: The Picard-Lindelöf Theorem

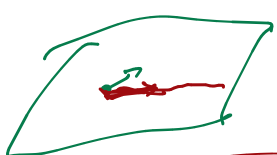
Facts to Know:

In differential geometry, we need to know certain quantity exists, and usually this would be equivalent to the existence to some ordinary differential equation. In differential equations, the Picard-Lindelöf theorem, also known as the Picard's existence theorem, Cauchy-Lipschitz theorem, or existence and uniqueness theorem gives a set of conditions under which an initial value problem has a unique solution.

For the equation $\boxed{y' = f(x,y)}$ with the initial value condition $y(\cancel{x_0}\, t_0) = y_0$, if the function $\boxed{f(x,y)}$ is continuous, and the partial derivative function $\boxed{f_y(x,y)}$ is also continuous ~~neat~~ near the point $(\cancel{x_0}\, t_0, y_0)$. Then there is an $\varepsilon > 0$, such that the solution exists on the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.

$f(x,y) = |x| \cdot y \qquad f_y(x,y) = |x|$

$$\begin{cases} y' = f(x,y) = |x|y \\ y(0) = 4 \end{cases}$$



$x' = \vec{f}(x,t). \qquad X(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

$\vec{f}(x,t)$ is a function $\vec{f}(x,t) = \vec{f}(x_1, \dots, x_n, t).$

Theorem Let $x' = \vec{f}(x,t)$, such that $x(t_0) = \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

① $\vec{f}(x,t)$ to be countinoues.

② When fixing t, $\vec{f}(x,\cdot)$ is differentable near $(b_1, \dots, b_n, t_0)$ $\Rightarrow$ $\exists\, \varepsilon > 0$. such that there is a unique solution $x = x(t)$ on $(t_0 - \varepsilon, t_0 + \varepsilon)$.

$$\begin{cases} y'' = f(t, y, y') \\ y(t_0) = a \\ y'(t_0) = b \end{cases}$$

$$x(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x'(t) = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ f(t, x_1, x_2) \end{bmatrix}$$

$$\begin{cases} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_2 \\ f(t, x_1, x_2) \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \end{cases}$$